Static shells for the Vlasov-Poisson and Vlasov-Einstein systems

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Abstract

We prove the existence of static, spherically symmetric solutions of the stellar dynamic Vlasov-Poisson and Vlasov-Einstein systems, which have the property that their spatial support is a finite, spherically symmetric shell with a vacuum region at the center.

1 Introduction

Large stellar systems such as galaxies or globular clusters can be described by a density function $f \geq 0$ on phase space. If collisions among the stars are neglected, f satisfies the so-called Vlasov or Liouville equation, which is then coupled to field equations for the gravitational interaction. Depending on whether one chooses a Newtonian or a general relativistic setting, the resulting nonlinear system of partial differential equations is the so-called Vlasov-Poisson or the Vlasov-Einstein system, respectively. In the present note we are interested in time independent spherically symmetric solutions of these systems. We call these solutions static, since due to the spherical symmetry their average velocity vanishes everywhere. The Vlasov-Poisson system then takes the following form:

$$v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \tag{1.1}$$

$$\frac{1}{r^2}(r^2 U')' = 4\pi\rho \tag{1.2}$$

where

$$\rho(r) = \rho(x) = \int f(x,v)\,dv. \tag{1.3}$$

Here $x, v \in \mathbb{R}^3$ denote position and momentum, $r = |x|$, $'$ denotes the derivative with respect to r, $f = f(x,v)$ must be spherically symmetric, i. e., $f(x,v) = f(Ax, Av)$ for every rotation $A \in \mathrm{SO}(3)$, $\rho(x) = \rho(r)$ denotes the spatial mass density of the ensemble and $U(x) = U(r)$ is the induced gravitational potential. We assume that all particles in the ensemble have the same mass which—like all other physical constants—is set to unity.

Under the corresponding assumptions the Vlasov-Einstein system takes the form

$$\frac{v}{\sqrt{1+v^2}} \cdot \partial_x f - \sqrt{1+v^2}\,\mu' \frac{x}{r} \cdot \partial_v f = 0, \tag{1.4}$$

$$e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi r^2 \rho, \tag{1.5}$$
$$e^{-2\lambda}(2r\mu' + 1) - 1 = 8\pi r^2 p, \tag{1.6}$$

where

$$\rho(r) = \rho(x) = \int \sqrt{1+v^2} f(x,v)\,dv, \tag{1.7}$$

$$p(r) = p(x) = \int \left(\frac{x \cdot v}{r}\right)^2 f(x,v) \frac{dv}{\sqrt{1+v^2}} \tag{1.8}$$

denote the spatial density of mass-energy and radial pressure, respectively. If $x = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ then the spacetime metric is given by

$$ds^2 = -e^{2\mu}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

As boundary conditions we require asymptotic flatness, i. e.,

$$\lim_{r \to \infty} \lambda(r) = \lim_{r \to \infty} \mu(r) = 0, \tag{1.9}$$

and a regular center, i. e.,

$$\lambda(0) = 0. \tag{1.10}$$

For the Vlasov-Poisson system the corresponding boundary condition is

$$\lim_{r \to \infty} U(r) = 0. \tag{1.11}$$

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All solutions of the above systems known so far have the property that the support of ρ contains a ball about the center; the only steady states where the support does not equal such a ball are the axially symmetric ones obtained in [6]. The purpose of the present note is to construct solutions whose support is a finite, spherically symmetric shell so that they have a vacuum region at the center. Given the fact that the dynamical behaviour of both the Vlasov-Poisson and the Vlasov-Einstein systems is far from being understood, static solutions with new structural properties are of interest in themselves. However, there is also a more specific motivation for the present investigation: In [11] the gravitational collapse of spherically symmetric solutions of the Vlasov-Einstein system is investigated numerically. Static solutions provide useful test cases for the corresponding numerical scheme. Since the center of symmetry is particularly difficult to handle, it is important to have static solutions both with matter and with vacuum at the center in order to assess the performance of the numerical scheme. As shown in [10] symmetric solutions of the time dependent problem which vanish near the center remain smooth. Thus, staying away from the center avoids analytic as well as numeric difficulties.

The way to construct such steady states is now described. Since the system is time independent, the particle energy must be a conserved quantity, since it is spherically symmetric, the same is true for the modulus of angular momentum. Indeed, the quantities

$$E = E(x,v) = \frac{1}{2}v^2 + U(x), \ L = L(x,v) = |x \times v|^2 \tag{1.12}$$

are constant along solutions of the characteristic equations

$$\dot{x} = v, \ \dot{v} = -\nabla U(x)$$

of the nonrelativistic Vlasov equation (1.1), and

$$E = E(x,v) = e^{\mu(r)}\sqrt{1 + v^2}, \ L = L(x,v) = |x \times v|^2 \tag{1.13}$$

are constant along characteristics of the relativistic Vlasov equation (1.4). Therefore, the ansatz

$$f(x,v) = \Phi(E,L) \tag{1.14}$$

satisfies the corresponding Vlasov equation and reduces the system to the field equation(s), where the source terms ρ or ρ and p now become functionals

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of U or μ, which are obtained by substituting the ansatz (1.14) into (1.3) or (1.7), (1.8) respectively. In passing we note that every static, spherically symmetric solution of the Vlasov-Poisson system must be of the form (1.14), cf. [1]. For the Vlasov-Einstein system this result, usually referred to as Jeans' Theorem, is not established.

One can easily see that ρ becomes a decreasing function of r if f is a function of the particle energy E only, the so-called isotropic case. Thus, to obtain a nontrivial solution with a vacuum region at the center, f must also depend on the angular momentum L, and it must vanish for L small, say for $L \leq L_0$ for some $L_0 > 0$. Once a solution of the field equation(s) is obtained, which has a vacuum region at the center, the main difficulty is to show that the support of the solution is actually bounded and the solution leads to a model with finite mass

$$M = \int \rho(x)\,dx < \infty;$$

in the case of the Vlasov-Einstein system this quantity is the so-called ADM mass. Finiteness of mass and support are obtained as follows. We take an ansatz function Φ, depending on the parameter L_0 in such a way that for $L_0 = 0$ known results give the existence of a solution with finite mass and finite support, in this case a ball about the center. Then a perturbation argument in L_0 is used to show that these properties persist also for $L_0 > 0$ but small. The smallness assumption on L_0 can then be removed by a scaling argument. The details of this procedure together with the precise statements of our results are given in the next section for the Vlasov-Poisson system, and in the last section for the Vlasov-Einstein system.

Before we go into this, we give a brief overview of the literature on the Vlasov-Poisson and the Vlasov-Einstein systems, starting with the former. We restrict ourselves to the stellar dynamics case; the plasma physics case, where the sign in the Poisson equation is reversed, is omitted. Global existence of classical solutions has been established in [4], cf. also [3, 12]. As far as the existence of stationary solutions of the Vlasov-Poisson system is concerned we mention [1, 2, 6]. The main result on the initial value problem for the Vlasov-Einstein system is a global existence theorem for small, spherically symmetric data [8]. Spherically symmetric steady states for the Vlasov-Einstein system are constructed in [5, 9].

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2 The nonrelativistic case

Throughout this section we fix two parameters $k, l \in \mathbb{R}$ with

$$k > -1, \ l > -1, \ k + l + \frac{1}{2} \geq 0, \ k < 3l + \frac{7}{2}.$$

We make the ansatz

$$f(x,v) = c_0 (E_0 - E)_+^k (L - L_0)_+^l \tag{2.1}$$

where E and L are defined as in (1.12), $(\cdot)_+$ denotes the positive part of the argument, and $c_0 > 0$, $E_0 < 0$, $L_0 \geq 0$. It is a straight forward computation to show that with this ansatz

$$\rho(r) = r^{2l} g\left(U(r) + \frac{L_0}{2r^2}\right), \tag{2.2}$$

where

$$g(u) := c_0 c_{kl} (E_0 - u)_+^{k+l+\frac{3}{2}}$$

and

$$c_{kl} := 2^{l+\frac{3}{2}} \pi \int_0^1 \frac{s^l}{\sqrt{1-s}} ds \int_0^1 s^{l+\frac{1}{2}} (1-s)^k ds,$$

and we have to solve

$$\frac{1}{r^2}(r^2 U')' = 4\pi r^{2l} g\left(U + \frac{L_0}{2r^2}\right), \ r > 0. \tag{2.3}$$

The exponents k and l are kept fixed, while the parameters c_0, E_0, and L_0 may vary during our argument. The following theorem is the main result of the present section:

Theorem 1 *Let $M > 0$ and $R_0 > 0$. Then there exists a static, spherically symmetric solution (f, ρ, U) of the Vlasov-Poisson system (1.1), (1.2), (1.3), where f and ρ depend on U via (2.1) and (2.2). $U \in C^2([0,\infty[) \cap C^2(\mathbb{R}^3)$ is a solution of (2.3) satisfying the boundary condition (1.11), $\rho \in C^1([0,\infty[) \cap C^1(\mathbb{R}^3)$ has total mass M, and $\operatorname{supp}\rho = [R_i, R_0]$ for some $R_i \in [0, R_0[$, where $R_i > 0$ provided $L_0 > 0$. Instead of prescribing $M > 0$ and $R_0 > 0$ one may also prescribe $M > 0$ and $R_i > 0$. If $L_0 = 0$ and $0 \neq l \leq 1/2$ then the asserted regularity holds only on $\mathbb{R}^3 \setminus \{0\}$.*

Note that we identify spherically symmetric functions of x with the corresponding functions of $r = |x|$.

Proof: Let us fix some E_0 and $c_0 = 1$, and consider $L_0 = 0$ first. Then for $U_0(0) < E_0$ prescribed there exists a unique solution U_0 of

$$\frac{1}{r^2}(r^2 U')' = 4\pi r^{2l} g(U),$$

cf. [1]; this solution need not satisfy the boundary condition (1.11), but we will take care of that later. By [1], U_0 induces a steady state with finite mass and finite support, which means that for some $R_0 > 0$ we have $U_0(r) > E_0$ for all $r \geq R_0$. For $L_0 > 0$ we define

$$U_{L_0}(r) = U_0(0), \ 0 \leq r \leq r_{L_0}$$

where

$$r_{L_0} := \sqrt{\frac{L_0}{2(E_0 - U_0(0))}};$$

note that $U_{L_0}(r) + \frac{L_0}{2r^2} > E_0$ on $]0, r_{L_0}[$, so the right hand side of the Poisson equation vanishes on that interval. Now extend this towards the right by the solution of (2.3) with $U_{L_0}(r_{L_0}) = U_0(0)$, $U'_{L_0}(r_{L_0}) = 0$. The latter exists on $[0, \infty[$, which can be shown using [1], but this also follows from the arguments below. Upon integrating the Poisson equation we find

$$U'_0(r) = \frac{4\pi}{r^2} \int_0^r s^{2+2l} g(U_0(s)) \, ds,$$

$$U'_{L_0}(r) = \frac{4\pi}{r^2} \int_0^r s^{2+2l} g\left(U_{L_0}(s) + \frac{L_0}{2s^2}\right) ds,$$

where the latter integral is zero for $r \leq r_{L_0}$. Now observe that $U_0(r) \geq U_0(0)$ and $U_{L_0}(r) \geq U_0(0)$ for $r \geq 0$, and on the set $[U_0(0), \infty[$ the function g is bounded and Lipschitz; recall that we assume $k + l + 1/2 \geq 0$. For $0 \leq r \leq r_{L_0}$ we have

$$\left|U'_0(r) - U'_{L_0}(r)\right| \leq \frac{C}{r^2} \int_0^r s^{2+2l} ds = Cr^{2l+1}$$

and for $r \geq r_{L_0}$

$$\left|U'_0(r) - U'_{L_0}(r)\right| \leq U'_0(r_{L_0}) + \frac{C}{r^2} \int_{r_{L_0}}^r s^{2+2l} \left(|U_0(s) - U_{L_0}(s)| + \frac{L_0}{2s^2}\right) ds$$

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$$\leq C r_{L_0}^{2l+1} + \frac{C}{r^2} L_0 r_{L_0}^{2\epsilon-2} \int_0^r s^{2+2l-2\epsilon} ds$$

$$+ \frac{C}{r^2} \int_{r_{L_0}}^r s^{2+2l} |U_0(s) - U_{L_0}(s)| ds$$

$$\leq C L_0^\epsilon r^{2l+1-2\epsilon} + \frac{C}{r^2} \int_{r_{L_0}}^r s^{2+2l} |U_0(s) - U_{L_0}(s)| ds,$$

where $\epsilon > 0$ is such that $1 + l - \epsilon > 0$; constants denoted by C may depend on U_0 and R_0, but not on r or L_0, and may change from line to line. Thus

$$|U_0(r) - U_{L_0}(r)| \leq C L_0^\epsilon + C \int_{r_{L_0}}^r \frac{1}{s^2} \int_{r_{L_0}}^s \sigma^{2+2l} |U_0(\sigma) - U_{L_0}(\sigma)| d\sigma\, ds$$

$$\leq C L_0^\epsilon + C \int_0^r s^{2l+1} \sup_{0 \leq \sigma \leq s} |U_0(\sigma) - U_{L_0}(\sigma)| ds,$$

and the latter inequality holds for $0 \leq r \leq R_0$. By Gronwall's Lemma,

$$|U_0(r) - U_{L_0}(r)| \leq C L_0^\epsilon,\ 0 \leq r \leq R_0;$$

note that s^{2l+1} is integrable over this interval. In particular, U_{L_0} must exist at least on this interval for L_0 small. We conclude that

$$U_{L_0}(R_0) > E_0$$

for $L_0 > 0$ sufficiently small, and by monotonicity,

$$U_{L_0}(r) + \frac{L_0}{2r^2} > E_0,\ r \geq R_0.$$

We fix a small $L_0 > 0$ and let $R_i = r_{L_0}$ and $U = U_{L_0}$ etc. Then U leads to a steady state with finite radius and finite mass. The regularity assertions are obvious in the case $L_0 > 0$. In the case $L_0 = 0$ the exponent l has to be restricted in such a way that $\rho'(0) = U'(0) = 0$ and $U''(0)$ exists.

Let $V(r) = U(r) + \frac{L_0}{2r^2}$. Then $V(R_i) = E_0$ and $V'(R_i) < 0$, whence $V(r) < E_0$ in a right neighborhood of R_i. Thus the induced mass density is nontrivial and R_i is the radius of the inner boundary of its support. If we define $M(r) = 4\pi \int_0^r s^2 \rho(s)\, ds$ then $M(r)$ is increasing with $M(r) > 0$ for $r > R_i$, and

$$V'(r) = \frac{M(r)}{r^2} - \frac{L_0}{r^3}.$$

Thus there can be at most one value of $r > R_i$ where $V'(r)$ changes sign, and if we define $R_0 = \inf\{r > R_i \,|\, V(r) = E_0\}$, this set being nonempty by what we showed above, then $R_0 > R_i$, and $\operatorname{supp}\rho = [R_i, R_0]$, i. e., the support of the steady state consists of a single shell and not several nested ones.

So far the boundary condition (1.11) need not be satisfied, but $U(\infty) = \lim_{r \to \infty} U(r) > E_0$ exists, and by slightly abusing notation we can redefine

$$U = U - U(\infty), \ \ E_0 = E_0 - U(\infty).$$

This leaves the distribution function f unchanged, and in addition (1.11) is now satisfied.

Finally we note that if f is a solution of the static Vlasov-Poisson system with mass M and $\operatorname{supp}\rho = [R_i, R_0]$ then the function

$$f_{\lambda,\mu}(x,v) = \gamma^3 \lambda^{-1} f(\gamma x, \gamma \lambda^{-1} v)$$

is also a solution, with mass $M(\lambda, \gamma) = \lambda^2 \gamma^{-3} M$ and support of the spatial density equal to $[R_i/\gamma, R_0/\gamma]$. Choosing γ and λ appropriately any prescribed value for R_0 (or R_i) and M can be obtained. Obviously, the constants c_0, E_0, and L_0 in the original ansatz (2.1) are changed by this scaling, but the boundary condition (1.11) is not.

3   The relativistic case

Throughout this section we fix two parameters $k, l \in \mathbb{R}$ with

$$k \geq 0, \ l > -\frac{1}{2}, \ k < 3l + \frac{7}{2}.$$

We again make the ansatz

$$f(x,v) = c_0 (E_0 - E)_+^k (L - L_0)_+^l \tag{3.1}$$

where E and L are now defined as in (1.13), and c_0, $E_0 > 0$, $L_0 \geq 0$. With this ansatz

$$\rho(r) = r^{2l} e^{-(2l+4)\mu} g\left(e^{\mu}\sqrt{1 + L_0/r^2}\right), \tag{3.2}$$

$$p(r) = r^{2l} e^{-(2l+4)\mu} h\left(e^{\mu}\sqrt{1 + L_0/r^2}\right), \tag{3.3}$$

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where

$$g(u) := c_0 c_l \int_u^\infty (E_0 - E)_+^k E^2 (E^2 - u^2)^{l+1/2} dE, \qquad (3.4)$$

$$h(u) := \frac{c_0 c_l}{2l+3} \int_u^\infty (E_0 - E)_+^k (E^2 - u^2)^{l+3/2} dE, \qquad (3.5)$$

and

$$c_l := 2\pi \int_0^1 \frac{s^l}{\sqrt{1-s}} ds.$$

Taking into account the boundary condition (1.10) we can integrate the field equation (1.5) to obtain

$$e^{-2\lambda} = 1 - \frac{8\pi}{r} \int_0^r s^2 \rho(s) ds,$$

and substituting this into (1.6) reduces the static, spherically symmetric Vlasov-Einstein system to the equation

$$\mu'(r) = \left(1 - \frac{8\pi}{r} \int_0^r s^2 \rho(s) ds\right)^{-1} \left(4\pi r p(r) + \frac{4\pi}{r^2} \int_0^r s^2 \rho(s) ds\right), \qquad (3.6)$$

where ρ and p are now functionals of μ given by (3.2), (3.3), (3.4), (3.5). The following theorem is the main result of the present section:

Theorem 2 *There exists a static, spherically symmetric solution $(f, \rho, p, \lambda, \mu)$ of the Vlasov-Einstein system (1.4), (1.5), (1.6), (1.7), (1.8), where f, ρ, and p depend on μ via (3.1), (3.2), and (3.3) in a neighborhood of their support. $\lambda, \mu \in C^2([0, \infty[) \cap C^2(\mathbb{R}^3)$ satisfy the boundary conditions (1.9), (1.10), and μ is a solution of (3.6). $\rho, p \in C^1([0, \infty[) \cap C^1(\mathbb{R}^3)$ with $\operatorname{supp}\rho = \operatorname{supp}p = [R_i, R_0]$ for some $0 \le R_i < R_0 < \infty$, where $R_i > 0$ provided $L_0 > 0$. The ADM mass M is finite, and one can prescribe $M > 0$ or $R_0 > 0$ or $R_i > 0$. If $L_0 = 0$ and $0 \ne l \le 1/2$ then the asserted regularity holds only on $\mathbb{R}^3 \setminus \{0\}$.*

Proof: Consider first the case $L_0 = 0$. As was shown in [5] there exists $E_0 > 0$ and a solution μ_0 of (3.6) with $e^{\mu_0(0)} < E_0$ and $e^{\mu_0(R_0)} > E_0$ for some $R_0 > 0$. We choose R_0 such that $E_0 < e^{\mu_0(R_0)} < E_0 + 1$. For any $L_0 > 0$ we define

$$r_{L_0} := \sqrt{\frac{L_0}{E_0^2 e^{-2\mu_0(0)} - 1}}.$$

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Then

$$e^{\mu_0(0)}\sqrt{1+L_0/r_{L_0}^2}=E_0,\ e^{\mu_0(0)}\sqrt{1+L_0/r^2}>E_0,\ r\in[0,r_{L_0}[,$$

which means that $\mu_{L_0}(r)=\mu_0(0)$ solves (3.6) on $[0,r_{L_0}]$ with $\rho_{L_0}(r)=p_{L_0}(r)=0$; in what follows ρ_{L_0} and p_{L_0} are always given in terms of μ_{L_0} by (3.2) and (3.3) respectively. By [5, Thm. 3.1] μ_{L_0} can be extended as a solution of (3.6) for $r\geq r_{L_0}$. We want to show that $\exp(\mu_{L_0}(R_0))>E_0$ for $L_0>0$ small so we may assume that $\mu_{L_0}(R_0)<\mu_0(R_0)+1$ since otherwise we are done. By monotonicity,

$$\mu_0(0)\leq\mu_0(r),\ \mu_{L_0}(r)<\mu_0(R_0)+1,\ r\in[0,R_0].$$

The functions g and h can be shown to be continuously differentiable, cf. [9, Lemma 2.1], and they vanish for $u>E_0$. Thus

$$|\rho_{L_0}(r)-\rho_0(r)|\leq Cr^{2l}\left|e^{-(2l+4)\mu_{L_0}}-e^{-(2l+4)\mu_0}\right|g\left(e^{\mu_{L_0}}\sqrt{1+L_0/r^2}\right)$$

$$+Cr^{2l}e^{-(2l+4)\mu_0}\left|g\left(e^{\mu_{L_0}}\sqrt{1+L_0/r^2}\right)-g(e^{\mu_0})\right|$$

$$\leq C\left(r^{2l}|\mu_{L_0}(r)-\mu_0(r)|+r^{2l-1}\sqrt{L_0}\right),\ r\in]0,R_0],\quad(3.7)$$

and similarly

$$|p_{L_0}(r)-p_0(r)|\leq C\left(r^{2l}|\mu_{L_0}(r)-\mu_0(r)|+r^{2l-1}\sqrt{L_0}\right),\ r\in]0,R_0].\qquad(3.8)$$

Constants denoted by C may depend on μ_0 and R_0, but never on r or L_0, and may change from line to line. From (3.6) we obtain the estimate

$$|\mu'_{L_0}(r)-\mu'_0(r)|\leq\left|\left(1-\frac{8\pi}{r}\int_0^r s^2\rho_{L_0}(s)\,ds\right)^{-1}-\left(1-\frac{8\pi}{r}\int_0^r s^2\rho_0(s)\,ds\right)^{-1}\right|$$

$$\left(4\pi r p_{L_0}(r)+\frac{4\pi}{r^2}\int_0^r s^2\rho_{L_0}(s)\,ds\right),$$

$$+4\pi\left(1-\frac{8\pi}{r}\int_0^r s^2\rho_0(s)\,ds\right)^{-1}$$

$$\left|r p_{L_0}(r)+\frac{1}{r^2}\int_0^r s^2\rho_{L_0}(s)\,ds-r p_0(r)-\frac{1}{r^2}\int_0^r s^2\rho_0(s)\,ds\right|$$

$$=I+II.$$

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Suppose that
$$\sup_{0 \leq r \leq R} |\mu_{L_0}(r) - \mu_0(r)| \leq \gamma \tag{3.9}$$

for some $\gamma > 0$ and $R \in]0, R_0]$. Then by (3.7),

$$\frac{8\pi}{r} \int_0^r s^2 |\rho_{L_0}(s) - \rho_0(s)| \, ds \leq \frac{C}{r} \int_0^r \left(s^{2l+2}\gamma + s^{2l+1}\sqrt{L_0} \right) ds \leq C_1 \left(\gamma + \sqrt{L_0} \right),$$

recall that $l > -1/2$. Now we choose $\gamma > 0$ such that

$$\sup_{0 < r \leq R_0} \frac{8\pi}{r} \int_0^r s^2 \rho_0(s) \, ds + 2C_1\gamma < 1.$$

For $L_0 \leq \gamma^2$ this implies that

$$\sup_{0 < r \leq R} \frac{8\pi}{r} \int_0^r s^2 \rho_{L_0}(s) \, ds \leq \sup_{0 < r \leq R} \frac{8\pi}{r} \int_0^r s^2 \rho_0(s) \, ds + 2C_1\gamma < 1.$$

Thus
$$\left(1 - \frac{8\pi}{r} \int_0^r s^2 \rho_{L_0}(s) \, ds \right)^{-1} < C, \ 0 < r \leq R,$$

provided (3.9) holds. This allows us to estimate the term I above:

$$I \leq Cr^{2l} \int_0^r s^2 |\rho_{L_0}(s) - \rho_0(s)| \, ds$$
$$\leq Cr^{4l+2}\sqrt{L_0} + Cr^{2l} \int_0^r s^{2l+2} |\mu_{L_0}(s) - \mu_0(s)| \, ds.$$

As to the second term we find that

$$II \leq Cr|p_{L_0}(r) - p_0(r)| + \frac{C}{r^2} \int_0^r s^2 |\rho_{L_0}(s) - \rho_0(s)| \, ds$$
$$\leq C\sqrt{L_0}r^{2l} + Cr^{2l+1} |\mu_{L_0}(r) - \mu_0(r)| + C \int_0^r s^{2l} |\mu_{L_0}(s) - \mu_0(s)| \, ds.$$

Thus
$$|\mu_{L_0}(r) - \mu_0(r)| \leq C \left(\sqrt{L_0} + \int_0^r s^{2l} |\mu_{L_0}(s) - \mu_0(s)| \, ds \right)$$

on $[0, R]$, and by Gronwall's lemma,

$$|\mu_{L_0}(r) - \mu_0(r)| \leq C\sqrt{L_0}, \ r \in [0, R].$$

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By choosing L_0 small we can make sure that (3.9) holds on $[0, R_0]$ so that the previous estimate holds on $[0, R_0]$ as well. In particular, $\exp(\mu_{L_0}(R_0)) > E_0$ provided L_0 is sufficiently small. We fix a sufficiently small L_0 and let $R_i = r_{L_0}$ and $\mu = \mu_{L_0}$ etc.

It is easy to see that $e^{\mu(r)}\sqrt{1 + L_0/r^2} < E_0$ and thus $\rho(r) > 0$ on some interval $]R_i, R[$. Take $R_0 > R_i$ the smallest such R with the property that $\rho = 0$ in a right neighbourhood of R_0. It is not clear that $e^{\mu(r)}\sqrt{1 + L_0/r^2} > E_0$ for all $r > R_0$, but we can simply extend the solution towards the right of R_0 by the corresponding vacuum solution. Thus, while f, ρ, and p depend on μ via (3.1), (3.2), and (3.3) in a neighborhood of their support, this need not be true for all values of r. Clearly, $\mu(\infty) = \lim_{r \to \infty} \mu(r)$ exists. If we redefine

$$\mu(r) = \mu(r) - \mu(\infty), \quad E_0 = E_0 e^{-\mu(\infty)}$$

we satisfy the boundary condition at infinity while the constant c_0 in (3.1) is multiplied by $e^{k\mu(\infty)}$.

Finally, if $f(x, v)$ defines a steady state, so does

$$f_a(x, v) = a^2 f(ax, v)$$

for any $a > 0$. The rescaled function f_a has spatial support $[a^{-1}R_i, a^{-1}R_0]$ and ADM mass

$$a^{-1} \int f(x, v)\sqrt{1 + v^2}\, dv\, dx$$

which shows that by rescaling a given solution we can get any prescribed value for the ADM mass, or the inner, or the outer radius.

Final remark: It would be desirable to have a complete parametrization of all steady states as constructed in Thms. 1 and 2 (for k and l fixed), that is to say, a result of the form: For every $0 \le R_i < R_0 < \infty$ and $M > 0$ there exists a unique steady state of the form (2.1) or (3.1) with support $[R_i, R_0]$ and mass M. Such a result can be obtained in the Vlasov-Poisson case for $L_0 = 0$, cf. [7].

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